Fading Communication:

In wireless communication the fading factor $\tilde{\alpha}$ is a random variable. And for the transmitted signal x the received signal is written as

$$y = \tilde{\alpha} \times x + n \tag{1}$$

where n is the AWGN and N_0 is the power spectral density (ω/Hz). The mean square of the RV $\tilde{\alpha}$ is denoted by

$$\Omega = \overline{\tilde{\alpha}^2} \tag{2}$$

And the probability density function of RV α is denoted by $p_{\tilde{\alpha}}(\alpha)$. The instantaneous SNR is defined as

$$\tilde{\gamma} = \tilde{\alpha}^2 \frac{E_s}{N_0} \tag{3}$$

where E_s is the energy per symbol.

Performance evaluation of digital communication systems, such as BER, over fading channels will generally be a function of the average SNR per symbol, i.e., $\bar{\gamma}$.

The amount of fading is defined as

$$AF = \frac{Var(\tilde{\alpha}^2)}{\left(E(\tilde{\alpha}^2)\right)^2} = \frac{E((\tilde{\alpha}^2 - \Omega)^2)}{\Omega^2} = \frac{E(\tilde{\gamma}^2) - \left(E(\tilde{\gamma})\right)^2}{\left(E(\tilde{\gamma})\right)^2} \tag{4}$$

For slowly varying channels the fading amplitude is constant over a symbol time and the PDF of the SNR is given as

$$p_{\tilde{\gamma}}(\gamma) = \frac{p_\alpha\left(\sqrt{\frac{\Omega\gamma}{\bar{\gamma}}}\right)}{2\sqrt{\frac{\Omega\gamma}{\bar{\gamma}}}}. \tag{5}$$

Rayleigh Fading:

For Rayleigh channels the fading amplitude is distributed according to

$$p_{\tilde{\alpha}}(\alpha) = \frac{2\alpha}{\Omega} \exp\left(-\frac{\alpha^2}{\Omega}\right) \tag{6}$$

Exercise: Show that

$$\int_0^\infty \alpha^2 p_{\tilde{\alpha}}(\alpha) d\alpha = \Omega$$

The pdf of the instantaneous SNR for the Rayleigh distributed channel is given as

$$p_{\tilde{r}}(\gamma) = \frac{1}{\bar{\gamma}} \exp\left(-\frac{\gamma}{\bar{\gamma}}\right).$$

The moments of this pdf are given as

$$E(\tilde{\gamma}^k) = \Gamma(1+k)\bar{\gamma}^k$$

where $\Gamma(\cdot)$ is the Gamma function.

Nakagami-q (Hoyt) Fading:

The pdf of the SNR per symbol for Hoyt channel is given as

$$p_{\tilde{q}}(\alpha) = \frac{(1+q^2)}{2q\bar{\gamma}} exp\left(-\frac{(1+q^2)^2\gamma}{4q^2\bar{\gamma}}\right) I_0\left(\frac{(1-q^4)\gamma}{4q^2\bar{\gamma}}\right) \quad \gamma \geq 0 \tag{7}$$

The moments of this pdf are given as

$$E(\tilde{\gamma}^k) = \Gamma(1+k)\ {}_2F_1\left(-\frac{k-1}{2}, -\frac{k}{2}; 1, \left(\frac{1-q^2}{1+q^2}\right)^2\right) \bar{\gamma}^k$$

where ${}_2F_1(.,.;.,.)$ is the hypergeometric function. The AF of the Hoyt distribution is given as

$$AF_q = \frac{2(1+q^4)}{(1+q^2)^2} \quad 0 \leq q \leq 1$$

Hoyt distribution reduces to one-sided Gaussian distribution when $q = 0$ and it reduces to Rayleigh distribution when $q = 1$.

There are also other distributions such as Nakagami-m (Rice), Nakagami-m, Log-Normal Shadowing, etc.

Average Error Probability of Binary Signals over Flat Fading Channels

The instantaneous bit error (BER) probability of coherent, differentially coherent, and non-coherent detection of binary transmitted signals over the AWGN channel is given by

$$P_b(E) = \frac{\Gamma\left(b, \frac{aE_b}{N_0}\right)}{2\Gamma(b)} \tag{8}$$

where $\Gamma(\cdot,\cdot)$ is the incomplete Gamma function defined as

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt \tag{9}$$

and $\Gamma(\cdot)$ is the Gamma function. Making change of variable $t = \frac{x}{\sin^2(\phi)}$ in (9) we get

$$\Gamma(\alpha, x) = 2x^\alpha \int_0^{\pi/2} \frac{\cos\phi}{(\sin\phi)^{2\alpha+1}} \exp\left(-\frac{\alpha\gamma}{sin^2\,\phi}\right) d\phi \tag{10}$$

And substituting (10) in (8) we get

$$P_b(\gamma) = \frac{(a\gamma)^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \exp\left(-\frac{a\gamma}{sin^2\,\phi}\right) d\phi \tag{11}$$

which is also called as conditional BER.

Let's derive (10) now. The variable change $t = \frac{x}{\sin^2(\phi)}$ can be written as $t = x\sin^{-2}(\phi)$ and taking differential of both sides we get

$$dt = -2x\sin^{-3}(\phi)\cos(\phi)\,d\phi$$

when placed in (9) we obtain

$$\Gamma(\alpha, x) = \int_{\pm\frac{\pi}{2}}^{0} e^{-\frac{x}{\sin^2(\phi)}} \left(\frac{x}{\sin^2\phi}\right)^{\alpha-1} - 2x\sin^{-3}(\phi)\cos(\phi)\,d\phi$$

which can be easily simplified as

$$\Gamma(\alpha, x) = 2x^\alpha \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2\alpha+1}} \exp\left(-\frac{x}{(\sin\phi)^2}\right) d\phi$$

where substituting $\alpha = b$, and $x = aE_b/N_0$we obtain

$$\Gamma(b, aE_b/N_0) = \frac{\left(\frac{aE_b}{N_0}\right)^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \exp\left(-\frac{\frac{aE_b}{N_0}}{(\sin\phi)^2}\right) d\phi$$

which can alternatively written as

Then the $P_b(E)$ expression in (8) can be written as

$$\Gamma(\gamma) = \frac{(a\gamma)^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\phi \tag{12}$$

Average BER over Fading Channels:

Averaging the instantaneous BER for a given SNR over all SNRs we obtain the average BER expression as

$$\bar{P}_b = \int_0^{\infty} P_b(\gamma) p_{\tilde{\gamma}}(\gamma) d\gamma \tag{13}$$

And this expression is evaluated for different fading channels, i.e., Rayleigh, Hoyt, Ricean, Log-Normal, etc.

Substituting (12) into (13) we get

$$\bar{P}_b = \int_0^{\infty} \frac{(a\gamma)^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\phi \, p_{\tilde{\gamma}}(\gamma) d\gamma$$

which can be arranged as

$$\bar{P}_b = \frac{a^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \left(\int_0^{\infty} \gamma^b p_{\tilde{\gamma}}(\gamma) \ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma \right) d\phi$$

where the inner integral expression can be denoted by

$$\mathcal{L}(\bar{\gamma}, a, b, \phi) = \int_0^{\infty} \gamma^b p_{\tilde{\gamma}}(\gamma) \ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

which can be interpreted as the Laplace transform of $\gamma^b p_{\tilde{\gamma}}(\gamma)$ evaluated at $-\frac{a}{(\sin\phi)^2}$.

Note: Laplace transform of the function $f(t)$ is

$$\mathcal{L}(f(t)) = F(s) = \int_0^{\infty} f(t) e^{-st} dt.$$

We will now calculate $\bar{P}_b$ first for Rayleigh distribution then for other distributions.

$\bar{P}_b$ for Rayleigh Distribution:

Let's first calculate

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b p_{\tilde{\gamma}}(\gamma)\ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

for Rayleigh distribution. It is calculated as follows

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b \frac{1}{\bar{\gamma}} \exp\left(-\frac{\gamma}{\bar{\gamma}}\right)\ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

$$= \int_0^\infty \frac{\gamma^b}{\bar{\gamma}} \exp\left(-\frac{\gamma}{\bar{\gamma}} - \frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

$$= \int_0^\infty \frac{\gamma^b}{\bar{\gamma}} \exp\left(-\gamma\left(\frac{1}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)\right) d\gamma$$

when compared to

$$\int_0^\infty x^v e^{-sx} dx = \frac{\Gamma(v+1)}{s^{v+1}}, \qquad s > 0\ \ v > -1$$

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \frac{\gamma^b}{\bar{\gamma}} \exp\left(-\gamma\left(\frac{1}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)\right) d\gamma$$

$$= \frac{\Gamma(b+1)}{\bar{\gamma}} \times \frac{1}{\left(\frac{1}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)^{b+1}}$$

$$= \frac{\Gamma(b+1)}{\bar{\gamma}\left(\frac{1}{\bar{\gamma}}\right)^{b+1}} \times \frac{1}{\left(1 + \frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{b+1}}$$

$$= \frac{\Gamma(b+1)(\bar{\gamma})^b}{\left(1 + \frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{b+1}}$$

when substituted in

$$\bar{P}_b = \frac{a^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \mathcal{L}(\bar{\gamma}, a, b, \phi) d\phi$$

we obtain

$$\bar{P}_b = \frac{a^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \frac{\Gamma(b+1)(\bar{\gamma})^b}{\left(1 + \frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{b+1}} d\phi \tag{14}$$

In (14) let $t = \left(1 + \frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{-1}$ then $t^{-1} = \left(1 + \frac{a\bar{\gamma}}{(\sin\phi)^2}\right)$ and

$$t = \frac{\sin^2\phi}{\sin^2\phi + a\bar{\gamma}} \rightarrow \frac{dt}{d\phi} = \frac{\left((2\sin\phi\cos\phi)(\sin^2\phi + a\bar{\gamma}) - (2\sin\phi\cos\phi)\sin^2\phi\right)}{(\sin^2\phi + a\bar{\gamma})^2}$$

$$\rightarrow \frac{dt}{d\phi} = \frac{(2\sin\phi\cos\phi)a\bar{\gamma}}{(\sin^2\phi + a\bar{\gamma})^2}$$

$$\rightarrow d\phi = \frac{(\sin^2\phi + a\bar{\gamma})^2}{(2\sin\phi\cos\phi)a\bar{\gamma}} \mathrm{dt}$$

substituting in (14)

$$\bar{P}_b = \frac{a^b}{\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \Gamma(b+1)(\bar{\gamma})^b t^{b+1} \frac{(\sin^2\phi + a\bar{\gamma})^2}{(2\sin\phi\cos\phi)a\bar{\gamma}} \mathrm{dt}$$

$$= \frac{a^b}{\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} \frac{(\sin^2\phi + a\bar{\gamma})^2}{2a(\sin\phi)^{2b+2}} \Gamma(b+1)(\bar{\gamma})^{b-1} t^{b+1} \mathrm{dt}$$

and using $\frac{a\bar{\gamma}}{(t^{-1}-1)} = (\sin\phi)^2$

$$= \frac{a^b}{\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} \frac{\left(\frac{a\bar{\gamma}}{(t^{-1}-1)} + a\bar{\gamma}\right)^2}{2a\left(\frac{a\bar{\gamma}}{(t^{-1}-1)}\right)^{b+1}} \Gamma(b+1)(\bar{\gamma})^{b-1} t^{b+1} \mathrm{dt}$$

$$= \frac{a^b}{\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} \frac{\left(\frac{a\bar{\gamma}}{(t^{-1}-1)} + a\bar{\gamma}\right)^2}{2a\left(\frac{a\bar{\gamma}}{(t^{-1}-1)}\right)^{b+1}} \Gamma(b+1)(\bar{\gamma})^{b-1} t^{b+1} \mathrm{dt}$$

$$= \frac{a^b}{\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} \frac{(1-t)^{b-1}}{(2a)(a\bar{\gamma})^{b-1}} \Gamma(\mathrm{b}+1)\bar{\gamma}^{\mathrm{b}-1}\mathrm{dt}$$

$$= \frac{1}{2\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} (1-t)^{b-1}\, \Gamma(\mathrm{b}+1)\mathrm{dt}$$

where using $\Gamma(\mathrm{b}+1) = b\Gamma(b)$ we get

$$\bar{P}_b = \frac{1}{2\Gamma(b)} \int_0^{(1+a\bar{\gamma})^{-1}} (1-t)^{b-1}\, b\Gamma(b)\mathrm{dt}$$

$$= \frac{b}{2} \int_0^{(1+a\bar{\gamma})^{-1}} (1-t)^{b-1}\, \mathrm{dt}$$

And we know that the incomplete Beta function is defined as

$$\mathrm{B}_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}dt$$

then $\bar{P}_b$result can be expressed as

$$\bar{P}_b = \frac{b}{2} \int_0^{(1+a\bar{\gamma})^{-1}} (1-t)^{b-1}\, \mathrm{dt}$$

$$= \frac{b}{2} \mathrm{B}_{(1+a\bar{\gamma})^{-1}}(1,b)$$

$\bar{P}_b$ for Nakagami-*q* (Hoyt):

Let’s first calculate

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b p_{\tilde{\gamma}}(\gamma) \;\; \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma \qquad (15)$$

for Hoyt distribution. Hoyt distribution is given as

$$p_{\tilde{\gamma}}(\gamma) = \frac{(1+q^2)}{2q\bar{\gamma}} exp\left(-\frac{(1+q^2)^2\gamma}{4q^2\bar{\gamma}}\right) I_0\left(\frac{(1-q^4)\gamma}{4q^2\bar{\gamma}}\right) \quad \gamma \geq 0. \qquad (16)$$

Substituting (16) into (15) we obtain

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b \frac{(1+q^2)}{2q\bar{\gamma}} \exp\left(-\frac{(1+q^2)^2\gamma}{4q^2\bar{\gamma}}\right) I_0\left(\frac{(1-q^4)\gamma}{4q^2\bar{\gamma}}\right) \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

$$= \frac{(1+q^2)}{2q\bar{\gamma}} \int_0^\infty \gamma^b \exp\left(-\left(\frac{(1+q^2)^2}{4q^2\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)\gamma\right) I_0\left(\frac{(1-q^4)\gamma}{4q^2\bar{\gamma}}\right) d\gamma \tag{17}$$

We have the property

$$\int_0^\infty t^\mu I_0(at) e^{-pt}\, dt = \frac{\Gamma(\mu+1)}{s^{\mu+1}} P_\mu^0\left(\frac{p}{s}\right); \quad \mu > -1, \ \ s = \sqrt{p^2 - a^2} \tag{18}$$

where $P_\mu^0(\cdot)$ is the Legendre function.

Let's use (18) in (17), for this purpose let

$$t = \gamma, \qquad \mu = b, \qquad a = \frac{(1-q^4)}{4q^2\bar{\gamma}}, \qquad p = \frac{(1+q^2)^2}{4q^2\bar{\gamma}} + \frac{a}{(\sin\phi)^2}$$

then (17) becomes as

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+q^2)}{2q\bar{\gamma}} \times \frac{\Gamma(b+1)}{s^{b+1}} \times P_b^0\left(\frac{p}{s}\right) \tag{19}$$

where using the property

$$P_\mu^0(z) = P_\mu(z) = {}_2F_1\left(-\mu, \mu+1; 1; \frac{1-z}{2}\right)$$

in (19) we obtain

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+q^2)}{2q\bar{\gamma}} \times \frac{\Gamma(b+1)}{s^{b+1}} \times {}_2F_1\left(-b, b+1; 1; \frac{1-p/s}{2}\right)$$

which is equal to

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+q^2)}{2q\bar{\gamma}} \times \frac{\Gamma(b+1)}{s^{b+1}} \times {}_2F_1\left(-b, b+1; 1; \frac{1-p/\sqrt{p^2-a^2}}{2}\right).$$

And

$$s^{b+1} = (p^2 - a^2)^{\frac{b+1}{2}}$$
$$= \left[\left(\frac{(1+q^2)^2}{4q^2\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)^2 - \left(\frac{(1-q^4)}{4q^2\bar{\gamma}}\right)^2\right]^{\frac{b+1}{2}}$$

$$= \frac{1}{\bar{\gamma}^{b+1}} \left[\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)^2 - \left(\frac{(1-q^4)}{4q^2} \right)^2 \right]^{\frac{b+1}{2}}$$

Hence,

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+q^2)}{2q} \times \frac{\Gamma(b+1)\bar{\gamma}^b}{\left[\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)^2 - \left(\frac{(1-q^4)}{4q^2} \right)^2 \right]^{\frac{b+1}{2}}}$$

$$\times\ {}_2F_1\left(-b, b+1; 1; \frac{1 - p/\sqrt{p^2 - a^2}}{2} \right)$$

where the expression $(1 - p/\sqrt{p^2 - a^2})/2$ can be simplified as

$$\frac{1}{2} - \frac{1}{2} \frac{\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)}{\sqrt{\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)^2 - \left(\frac{(1-q^4)}{4q^2} \right)^2}}$$

Hence $\mathcal{L}(\bar{\gamma})$ turns out to be

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+q^2)}{2q} \times \frac{\Gamma(b+1)\bar{\gamma}^b}{\left[\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)^2 - \left(\frac{(1-q^4)}{4q^2} \right)^2 \right]^{\frac{b+1}{2}}}$$

$$\times\ {}_2F_1\left(-b, b+1; 1; \frac{1}{2} - \frac{1}{2} \frac{\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)}{\sqrt{\left(\frac{(1+q^2)^2}{4q^2} + \frac{a\bar{\gamma}}{(\sin\phi)^2} \right)^2 - \left(\frac{(1-q^4)}{4q^2} \right)^2}} \right)$$

$\bar{P}_b$ for Nakagami-*n* (Rice):

Rice distribution is given as

$$p_{\tilde{\gamma}}(\gamma) = \frac{(1+n^2)e^{-n^2}}{\bar{\gamma}} exp\left(-\frac{(1+n^2)\gamma}{\bar{\gamma}}\right) I_0\left(2n\sqrt{\frac{(1+n^2)\gamma}{\bar{\gamma}}}\right) \quad \gamma \geq 0. \tag{20}$$

Substituting (20) into

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b p_{\tilde{\gamma}}(\gamma) \ \ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

we obtain

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b \frac{(1+n^2)e^{-n^2}}{\bar{\gamma}} exp\left(-\frac{(1+n^2)\gamma}{\bar{\gamma}}\right) I_0\left(2n\sqrt{\frac{(1+n^2)\gamma}{\bar{\gamma}}}\right) \ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

which can be rearranged as

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+n^2)e^{-n^2}}{\bar{\gamma}} \int_0^\infty \gamma^b exp\left(-\left(\frac{(1+n^2)}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)\gamma\right) I_0\left(2\sqrt{\frac{n^2(1+n^2)\gamma}{\bar{\gamma}}}\right) \ d\gamma. \tag{21}$$

We have the property

$$\int_0^\infty t^{\mu-\frac{1}{2}} I_0(2\sqrt{at}) e^{-pt} dt = \frac{\Gamma\left(\mu+\frac{1}{2}\right) e^{\frac{a}{2p}}}{\sqrt{a}p^\mu} M_{-\mu,0}\left(\frac{a}{p}\right); \ \ \mu > -1/2 \tag{22}$$

Comparing (21) to (22) we have

$$t = \gamma, \qquad \mu - \frac{1}{2} = b, \qquad p = \left(\frac{(1+n^2)}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right), \quad a = \frac{n^2(1+n^2)}{\bar{\gamma}}$$

and using the property (22) we can write the Formula (21) as

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+n^2)e^{-n^2}}{\bar{\gamma}} \frac{\Gamma(b+1) e^{\frac{\frac{n^2(1+n^2)}{\bar{\gamma}}}{2\left(\frac{(1+n^2)}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)}}}{\sqrt{\frac{n^2(1+n^2)}{\bar{\gamma}}} \left(\frac{(1+n^2)}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)^{b+\frac{1}{2}}} M_{-b-\frac{1}{2},0}\left(\frac{\frac{n^2(1+n^2)}{\bar{\gamma}}}{\left(\frac{(1+n^2)}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)}\right)$$

which can be simplified as

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+n^2)e^{-n^2}\bar{\gamma}^b\Gamma(b+1)}{\sqrt{n^2(1+n^2)}} \frac{e^{\frac{n^2(1+n^2)}{2\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}}}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{b+\frac{1}{2}}} M_{-b-\frac{1}{2},0}\left(\frac{n^2(1+n^2)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}\right) \tag{23}$$

and recognizing the relation

$$M_{\mu,0}(z) = \sqrt{z}e^{-\frac{z}{2}}\ {}_1F_1(-\mu+\frac{1}{2};\ 1;\ z)$$

For (23) we have

$$z = \frac{n^2(1+n^2)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)},\ \ \mu = -b-\frac{1}{2}$$

The formula (23) can be written as

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+n^2)e^{-n^2}\bar{\gamma}^b\Gamma(b+1)}{\sqrt{n^2(1+n^2)}} \frac{e^{\frac{n^2(1+n^2)}{2\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}}}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{b+\frac{1}{2}}} \sqrt{\frac{n^2(1+n^2)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}}$$
$$e^{-\frac{\frac{n^2(1+n^2)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}}{2}}\ {}_1F_1(b+1;\ 1;\ \frac{n^2(1+n^2)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}) \tag{24}$$

which can be simplified as

$$\mathcal{L}(\bar{\gamma}) = \frac{(1+n^2)e^{-n^2}\bar{\gamma}^b\Gamma(b+1)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)^{b+1}}\ {}_1F_1(b+1;\ 1;\ \frac{n^2(1+n^2)}{\left((1+n^2)+\frac{a\bar{\gamma}}{(\sin\phi)^2}\right)}) \tag{25}$$

$\bar{P}_b$ for Nakagami-*m* Fading:

Nakagami-m distribution is defined as

$$p_{\tilde{\gamma}}(\gamma) = \frac{m^m\gamma^{m-1}}{\bar{\gamma}^m\Gamma(m)}\exp\left(-m\frac{\gamma}{\bar{\gamma}}\right). \tag{26}$$

Substituting (26) into

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b p_{\tilde{\gamma}}(\gamma) \ \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

we obtain

$$\mathcal{L}(\bar{\gamma}) = \int_0^\infty \gamma^b \frac{m^m \gamma^{m-1}}{\bar{\gamma}^m \Gamma(m)} \exp\left(-m\frac{\gamma}{\bar{\gamma}}\right) \exp\left(-\frac{a\gamma}{(\sin\phi)^2}\right) d\gamma$$

which can be rearranged as

$$\mathcal{L}(\bar{\gamma}) = \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \int_0^\infty \gamma^{b+m-1} \exp\left(-\left(\frac{m}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)\gamma\right) d\gamma \tag{27}$$

and comparing the Laplace transform

$$\int_0^\infty x^v e^{-sx} dx = \frac{\Gamma(v+1)}{s^{v+1}}$$

to (27) we have

$$x = \gamma, \quad v = b + m - 1, \quad s = \left(\frac{m}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right).$$

Then (27) can be evaluated as

$$\mathcal{L}(\bar{\gamma}) = \frac{m^m}{\bar{\gamma}^m \Gamma(m)} \frac{\Gamma(b+m)}{\left(\frac{m}{\bar{\gamma}} + \frac{a}{(\sin\phi)^2}\right)^{b+m}}$$

which can be written as

$$\mathcal{L}(\bar{\gamma}) = \frac{\Gamma(b+m)}{\Gamma(m)} \frac{\left(\frac{\bar{\gamma}}{m}\right)^b}{\left(1 + \frac{a\bar{\gamma}}{m(\sin\phi)^2}\right)^{b+m}} \tag{28}$$

When (28) is substituted into

$$\bar{P}_b = \frac{a^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2b+1}} \mathcal{L}(\bar{\gamma}, a, b, \phi) d\phi$$

we obtain

$$\bar{P}_b = \frac{a^b}{\Gamma(b)} \int_0^{\frac{\pi}{2}} \frac{\cos(\phi)}{(\sin\phi)^{2\mathrm{b}+1}} \frac{\Gamma(b+m)}{\Gamma(m)} \frac{\left(\frac{\bar{\gamma}}{m}\right)^b}{\left(1+\frac{a\bar{\gamma}}{m(\sin\phi)^2}\right)^{b+m}} d\phi$$

And making the change of variable

$$t = \left(1+\frac{a\bar{\gamma}}{m(\sin\phi)^2}\right)^{-1}$$